Journal of Mathematical Sciences: Advances and Applications
Volume 25, 2014, Pages 57-61

ON THE SPECTRUM OF A THREE-PARTICLE MODEL OPERATOR

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Abstract

We consider a model operator H associated with the system of three particles on a d-dimensional lattice. We obtain an analogue of the Faddeev equation for the eigenfunctions of H and also we describe the spectrum of H.

Many works are devoted to the investigation of the essential spectrum of the Schrödinger operators on a lattice, see, e.g., [1-3]. In particular, in [1], it was proved that the essential spectrum of a three-particle Schrödinger operator on a lattice is the union of at most finitely many closed intervals even in the case, where the corresponding two-particle Schrödinger operator on a lattice has an infinite number of eigenvalues.

In the present paper, we consider a model operator H associated to a system of three particles on a d-dimensional lattice, where the role of a two-particle Schrödinger operator on a lattice played by the Friedrichs model. We discuss the case where the kernel of perturbation operator (partial integral operator) has rank n with $n \geq 3$. We describe the

2010 Mathematics Subject Classification: Primary 81Q10; Secondary 35P20, 47N50.

Keywords and phrases: three-particle model operator, Hilbert-Schmidt class, Faddeev equation, spectrum.

Received December 30, 2013

essential and discrete spectrum of H and also obtained an analogue of the Faddeev equation for the eigenfunctions of H. We note that the operator H can be considered as a non-compact perturbation of the operators investigated in [4, 5]. In [4], the case $n = 1$ was considered and the case $n = 2$ was discussed in [5].

Denote by $\mathbb{T}^d$ the d-dimensional torus, the cube $(-\pi, \pi]^d$ with appropriately identified sides equipped with its Haar measure and by $L_2^s((\mathbb{T}^d)^2)$ the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbb{T}^d)^2$.

Let us consider an operator H acting on the Hilbert space $L_2^s((\mathbb{T}^d)^2)$:

$$(Hf)(p, q) := w(p, q)f(p, q) - \sum_{i=1}^{n}\left[v_i(p)\int_{\mathbb{T}^d} v_i(s)f(s, q)ds + v_i(q)\int_{\mathbb{T}^d} v_i(s)f(p, s)ds\right],$$

where $f \in L_2^s((\mathbb{T}^d)^2)$, the number n is a positive integer with $n \geq 3$, the functions $v_i(\cdot)$, $i = 1, \ldots, n$ are real-valued continuous functions on $\mathbb{T}^d$ and the function $w(\cdot, \cdot)$ is the real-valued symmetric continuous function on $(\mathbb{T}^d)^2$.

Under these assumptions, the operator H is bounded and self-adjoint.

To formulate main results of the paper, we introduce a family of bounded self-adjoint operators (Friedrichs models) $h(p)$, $p \in \mathbb{T}^d$, which acts in $L_2(\mathbb{T}^d)$ as

$$(h(p)f)(q) := w(p, q)f(q) - \sum_{i=1}^{n} v_i(q)\int_{\mathbb{T}^d} v_i(s)f(s)ds.$$

We denote by $\sigma(\cdot)$, $\sigma_{\text{ess}}(\cdot)$, and $\sigma_{\text{disc}}(\cdot)$ the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator, respectively, and set

$$m := \min_{p,q\in\mathbb{T}^d} w(p, q), \quad M := \max_{p,q\in\mathbb{T}^d} w(p, q), \quad \Sigma := \bigcup_{p\in\mathbb{T}^d} \sigma_{\text{disc}} h(p) \cup [m; M];$$

$$L_2^{(n)}(\mathbb{T}^d) := \{g = (g_1, \dots, g_n) : g_i \in L_2(\mathbb{T}^d), \quad i = 1, \dots, n\}.$$

Let $\mathbb{C}$ be the field of complex numbers. For each $z \in \mathbb{C} \setminus [m; M]$, we define the $n \times n$ block operator matrices $A(z)$ and $K(z)$ acting in the Hilbert space $L_2^{(n)}(\mathbb{T}^d)$ as

$$A(z) := (A_{ij}(z))_{i,j=1}^n, \qquad K(z) := (K_{ij}(z))_{i,j=1}^n,$$

where the operators $A_{ij}(z)$ are the multiplication operators by the function

$$a_{ij}(p; z) := \delta_{ij} - \int_{\mathbb{T}^d} \frac{v_i(s)v_j(s)ds}{w(p, s) - z}, \qquad \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and the operators $K_{ij}(z)$ are the integral operators with the kernel

$$K_{ij}(p, s; z) := \frac{v_j(p)v_i(s)}{w(p, s) - z}.$$

We note that for each $z \in \mathbb{C} \setminus [m; M]$, the operators $K_{ij}(z)$ belong to the Hilbert-Schmidt class and therefore $K(z)$ is a compact operator.

Lemma 1. *The operator $A(z)$, $z \in \mathbb{C} \setminus [m; M]$ is bounded and invertible if and only if $z \in \mathbb{C} \setminus \Sigma$.*

Now we give the main results of the paper.

The following theorem is an analogue of the well-known Faddeev's result for the operator H and establishes a connection between eigenvalues of H and $T(z) := A^{-1}(z)K(z)$.

Theorem 1. *The number* $z \in \mathbb{C} \setminus \Sigma$ *is an eigenvalue of the operator* H *if and only if the number* $\lambda = 1$ *is an eigenvalue of the operator* $T(z)$.

We point out that the equation $T(z)g = g$ is an analogue of the Faddeev type system of integral equations for eigenvectors of the operator H and its played crucial role in the analysis of the spectrum of H.

Since for any $z \in \mathbb{C} \setminus \Sigma$, the kernels of the entries of $T(z)$ are continuous functions on $(\mathbb{T}^d)^2$, the Fredholm determinant $\Delta(z)$ of the operator $I - T(z)$, where I is the identity operator in $L_2^{(n)}(\mathbb{T}^d)$, exists and is a real-analytic function on $\mathbb{C} \setminus \Sigma$.

According to Fredholm's theorem, the following lemma holds:

Lemma 2. *The number* $z \in \mathbb{C} \setminus \Sigma$ *is an eigenvalue of* H *if and only if* $\Delta(z) = 0$, *that is,*

$$\sigma_{\mathrm{disc}}(H) = \{z \in \mathbb{C} \setminus \Sigma : \Delta(z) = 0\}.$$

The following theorem describes the essential spectrum of the operator H.

Theorem 2. *For the essential spectrum of* H, *the equality* $\sigma_{\mathrm{ess}}(H) = \Sigma$ *holds*.

Sketch of proof of Theorem 2. By the definition of the essential spectrum, it is easy to show that $\Sigma \subset \sigma_{\mathrm{ess}}(H)$. Since the function $\Delta(\cdot)$ is analytic in $\mathbb{C} \setminus \Sigma$ by Lemma 2, we conclude that the set $\sigma(H) \setminus \Sigma = \{z \in \mathbb{C} \setminus \Sigma : \Delta(z) = 0\}$ is discrete. Thus $\sigma(H) \setminus \Sigma \subset \sigma(H) \setminus \sigma_{\mathrm{ess}}(H)$. Therefore, the inclusion $\sigma_{\mathrm{ess}}(H) \subset \Sigma$ holds.

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